Corrigé du Test 2.

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1) Soient $a,b>0$.

$$\ln\left(\frac{a^x+b^x}{2}\right) = \ln\left(\frac{1}{2}\left(e^{x\ln(a)}+e^{x\ln(b)}\right)\right)$$

$$= \ln\left(1+\frac{x}{2}\left(\ln a+\ln b\right)+o(x)\right)$$

$$\underset{0}{\sim} \frac{x}{2}\ln(ab)$$

Donc
$$\left(\frac{a^x+b^x}{2}\right)^{1/x} = \exp\left(\frac{1}{x}\ln\left(\frac{a^x+b^x}{2}\right)\right)$$

$$= \exp\left(\frac{1}{x}\left(\frac{x}{2}\ln(ab)+\underset{x\to0}{o}(x)\right)\right)$$

$$= \exp\left(\frac{\ln(ab)}{2}+\underset{x\to0}{o}(1)\right)$$

Ainsi
$$\boxed{\left(\frac{a^x+b^x}{2}\right)^{1/x} \xrightarrow[x\to0]{} \exp\left(\frac{\ln(ab)}{2}\right) = \sqrt{ab}}$$

2) Rappelons que
$$\begin{cases} \sin(x) = x-\dfrac{x^3}{6}+o(x^3) \\ \tan(x) = x+\dfrac{x^3}{3}+o(x^3) \end{cases}$$

donc
$$\exp(\sin x) = \exp(x)\exp\left(-\frac{x^3}{6}+o(x^3)\right)$$

$$= e^x\left(1-\frac{x^3}{6}+o(x^3)\right)$$

De même
$$\exp(\tan x) = e^x\left(1+\frac{x^3}{3}+o(x^3)\right)$$

Ainsi
$$\boxed{\frac{\exp(\sin x)-\exp(\tan x)}{\sin x-\tan x} = e^x\,\frac{-\frac{x^3}{6}-\frac{x^3}{3}+o(x^3)}{-\frac{x^3}{6}-\frac{x^3}{3}+o(x^3)} \xrightarrow[x\to0]{} 1}$$

3) $\ln(x+\sqrt{1+x^2}) - \ln x = \ln\left(1+\sqrt{1+\frac{1}{x^2}}\right)$

$$= \ln\left(2 + \left(\sqrt{1+\frac{1}{x^2}} - 1\right)\right)$$

$$= \ln 2 + \ln\left(1 + \underbrace{\frac{1}{2}\left(\sqrt{1+\frac{1}{x^2}} - 1\right)}\right)$$

$$= \frac{1}{4x^2} - \frac{1}{16x^4} + o\left(\frac{1}{x^4}\right)$$

Ainsi $\ln\left(1+\frac{1}{2}\left(\sqrt{1+\frac{1}{x^2}}-1\right)\right) = \frac{1}{4x^2} - \frac{3}{32x^4} + o\left(\frac{1}{x^4}\right)$

En conclusion,

$$\boxed{\ln(x+\sqrt{1+x^2}) - \ln x = \ln 2 + \frac{1}{4x^2} - \frac{3}{32x^4} + o\left(\frac{1}{x^4}\right)}$$

4) On a $\frac{2x}{x^2-1} = \frac{2x}{x^2\left(1-\frac{1}{x^2}\right)} = \frac{2}{x}\left(1+\frac{1}{x^2}+o\left(\frac{1}{x^2}\right)\right)$

$$= \frac{2}{x} + o\left(\frac{1}{x}\right)$$

Il vient

$$x\exp\left(\frac{2x}{x^2-1}\right) = x\left(1+\frac{2}{x}+o\left(\frac{1}{x}\right)\right)$$

$$= x+2+o(1)$$

Donc $y=x+2$ est asymptote en $+\infty$ à la courbe représentative de $x \mapsto x e^{\frac{2x}{x^2-1}}$.

5a) voir exemple III.C.1, chapitre "Séries"

b) idem

c) $$u_{n+1}^{-2} - u_n^{-2} = \sin(u_n)^{-2} - u_n^{-2}$$

$$= \left(u_n - \frac{u_n^3}{6} + \frac{u_n^5}{120} + o(u_n^5)\right)^{-2} - u_n^{-2}$$

$$= u_n^{-2}\left(\left(1 - \frac{u_n^2}{6} + \frac{u_n^4}{120} + o(u_n^4)\right)^{-2} - 1\right)$$

$$= u_n^{-2}\left(\cancel{1} - 2\left(-\frac{u_n^2}{6} + \frac{u_n^4}{120}\right) + 3\left(-\frac{u_n^2}{6}\right)^2 + o(u_n^4) \cancel{-1}\right)$$

$$= u_n^{-2}\left(\frac{1}{3}u_n^2 + \frac{1}{15}u_n^4 + o(u_n^4)\right)$$

$$= \frac{1}{3} + \frac{1}{15}u_n^2 + o(u_n^2)$$

Mais $u_n \underset{+\infty}{\sim} \sqrt{\frac{3}{n}}$ donc

$$u_{n+1}^{-2} - u_n^{-2} = \frac{1}{3} + \frac{1}{15}\cdot\frac{3}{n} + o\left(\frac{1}{n}\right) = \frac{1}{3} + \frac{1}{5n} + o\left(\frac{1}{n}\right)$$

Ainsi: $u_{n+1}^{-1} - u_n^{-2} - \frac{1}{3} \underset{+\infty}{\sim} \frac{1}{5n} \geq 0$. De plus $\sum_{n\geq 1} \frac{1}{n}$ diverge donc

par théorème de sommation,

$$\sum_{k=1}^{n-1}\left(u_{k+1}^{-2} - u_k^{-2} - \frac{1}{3}\right) \underset{+\infty}{\sim} \sum_{k=1}^{n-1} \frac{1}{5k} \sim \frac{1}{5}\ln(n-1)$$

Donc $\underbrace{u_n^{-2} - u_0^{-2}}_{=o(n)} - \frac{n-1}{3} \underset{+\infty}{\sim} \frac{1}{5}\ln(n-1) = \frac{1}{5}\Big(\ln(n) + \underbrace{\ln\left(1-\frac{1}{n}\right)}_{\xrightarrow[n\to+\infty]{} 0}\Big)$

Ainsi $u_n^{-2} - \frac{n}{3} \sim \frac{1}{5}\ln(n)$ et $u_n^{-2} = \frac{n}{3} + \frac{\ln n}{5} + o(\ln n)$

$$u_n = \sqrt{\frac{1}{u_n^{-2}}} = \sqrt{\frac{1}{\frac{n}{3} + \frac{\ln n}{5} + o(\ln n)}}$$

$$= \sqrt{\frac{3}{n}} \sqrt{\frac{1}{1+\frac{3\ln(n)}{5n}+o\left(\frac{\ln n}{n}\right)}}$$

$$= \sqrt{\frac{3}{n}} \left(1 - \frac{3\ln(n)}{10n} + o\left(\frac{\ln n}{n}\right)\right)$$

$$\boxed{u_n = \sqrt{\frac{3}{n}} - \frac{3\sqrt{3}\ln(n)}{10n\sqrt{n}} + o\left(\frac{\ln n}{n\sqrt{n}}\right)}$$

6) $$\chi_A = \begin{vmatrix} X-3 & 0 & 1 \\ -2 & X-4 & -2 \\ 1 & 0 & X-3 \end{vmatrix} = (X-4)\begin{vmatrix} X-3 & 0 & 1 \\ -2 & 1 & -2 \\ 1 & 0 & X-3 \end{vmatrix}$$

$$\underset{C_1 \leftarrow C_1 - C_3}{=} (X-4)\begin{vmatrix} X-4 & 0 & 1 \\ 0 & 1 & -2 \\ -X+4 & 0 & X-3 \end{vmatrix}$$

$$= (X-4)^2 \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 0 & X-3 \end{vmatrix} = (X-4)^2\left((X-3)+1\right)$$

$\chi_A = (X-4)^2(X-2)$ Ainsi $\boxed{Sp(A) = \{2,4\}}$

De plus

- $X \in E_2(A) \iff \begin{cases} 3x - z = 2x \\ 2x+4y+2z = 2y \\ -x+3z = 2z \end{cases}$

$$\iff \begin{cases} x = z \\ y = -2x \end{cases}$$

donc $\boxed{E_2(A) = \mathrm{Vect}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}}$

- $X \in E_4(A) \iff \begin{cases} 3x-z = 4x \\ 2x+4y+2z = 4y \\ -x+3z = 4z \end{cases}$

$$\Leftrightarrow \begin{cases} x = -z \\ y = y \end{cases}$$

Ainsi $\boxed{E_4(A) = \mathrm{Vect}\left(\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)}$

Ainsi $\boxed{\text{A est diagonalisable}}$.